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Characterizing the effects of self- and cross-diffusion on stationary patterns of a predator-prey system

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In this paper, we study a nonlinear coupled predator-prey diffusion system which widely exists in ecosystem. It is found that the self-diffusion and cross-diffusion do not change the stability of the semi equilibrium point of the corresponding predator-prey system. However, the two kinds of diffusion play an important role on the positive equilibrium, in virtue of which Turing instability of the corresponding diffusion system either continues to exist or disappears and becomes stable. On the stationary patterns of the nonlinear coupled system, we find some interesting results which differs from the phenomenon found in corresponding diffusion system. Strong cross-diffusion can make the corresponding system generate stationary patterns. Finally, numerical simulation is also done to verify the existence of the effects of self-diffusion and cross-diffusion.

Keywords: Turing instability; self-diffusion; cross-diffusion; stationary patterns

1. Introduction

Due to the universal existence of energy transformation, predator-prey system is very important in describing the population evolution[Murray, 1993]. In view of the differences in capturing food and consuming energy, a major trend in theoretical work on predator-prey dynamics has been launched so as to analysis more realistic models and functional responses, for example, Holling type[Peng & Wang, 2005; Shi *et al.*, 2010], Ivlev type[Kooij & Zegeling, 1996], Beddington-DeAangelis type[Beddington, 1975; Cantrell &

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Cosner, 2001; DeAngelis *et al.*, 1975], ratio-dependent type[Wang, 2004], and so on. In order to model the predator-prey mite outbreak interactions on fruit trees, Wollkind et al. [Wollkind & Logan, 1978; Wollkind *et al.*, 1988] adapted the following ordinary differential equations based on the model due to May[May, 1978],

$$\begin{cases} \dfrac{du}{dt} = ru\left(1 - \frac{u}{K}\right) - p(u)v, \\ \dfrac{dv}{dt} = v\left[s\left(1 - \frac{hv}{u}\right)\right], \end{cases} \tag{1}$$

where u, v represent the population densities of the prey and predator, respectively. In system (1), it is assumed that the prey grows logistically with carrying capacity K and intrinsic growth rate r in the absence of predation. The predator consumes the prey according to the functional response $p(u)$ which is only prey-dependent and grows logistically with intrinsic growth rate s and carrying capacity proportional to the population size of prey. The parameter h is the numbers of prey required to support one predator at equilibrium when v equals to u/h. The term $\frac{hv}{u}$ is called the Leslie-Gower term[Leslie & Gower, 1960],which measures the loss in predator population due to the rarity of its favorite food u.

In general, the response functions have an important effect on the dynamical behavior of predator-prey models. Particularly, the Beddington-DeAngelis functional response, $p(u,v) = \frac{ku}{a+bu+cv}$, is similar to Holling type II functional response but has an extra term cv in the denominator modelling mutual interference among predators. Hence, this kind of functional response performs even better than Holling type II functional response which is affected by both predator and prey. Meanwhile, the Beddington-DeAngelis functional response can be generated by a number of natural mechanisms because it admits rich but biologically reasonable dynamics (see [Beddington, 1975; DeAngelis *et al.*, 1975; Fan & Wang, 2009; Xiang *et al.*, 2013]).

Incorporating the Beddington-DeAngelis functional response into (1), we get

$$\begin{cases} \dfrac{du}{dt} = ru(1 - \frac{u}{K}) - \frac{kuv}{a+bu+cv}, \\ \dfrac{dv}{dt} = v\left[s\left(1 - \frac{hv}{u}\right)\right]. \end{cases} \tag{2}$$

In view of ecological aspect, system (2) only reflects population changes due to predation in a situation where predator and prey densities are not spatially dependent. It does not take into account either the fact that population is usually not homogeneously distributed, nor the fact that predators and prey naturally develop strategies for survival. Both of these considerations involve diffusion processes which can be quite intricate since different concentration levels of prey and predators cause different population movements[Liu *et al.*, 2018; Zhang & Zhu, 2019]. Such movements, that is, diffusion, self-diffusion and cross-diffusion, can be determined by the concentration changes of the species in along some spatial direction. The term diffusion describes the migration of species to avoid crowds produced by the population pressure due to the mutual interference between the individuals. Self-diffusion implies the movement of individuals from a higher concentration region to a lower one. Cross-diffusion, however, expresses the population fluxes of one species due to the presence of the other species, of which the coefficient can vary from positive, negative to zero. Usually, the positive one denotes the movement of the species in the direction of lower concentration of another species, while the negative one denotes that one species tend to diffuse in the direction of higher concentration of another species[Dubey *et al.*, 2001; Wen & Fu, 2009].

Along this line, the nonlinear coupled system of (2) can be written as the following model

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta[(d_1 + a_{11}u + a_{12}v)u] = ru(1 - \frac{u}{K}) - \frac{kuv}{a+bu+cv}, & t > 0,\ x \in \Omega, \\ \dfrac{\partial v}{\partial t} - \Delta[(d_2 + a_{21}u + a_{22}v)v] = v\left[s\left(1 - \frac{hv}{u}\right)\right], & t > 0,\ x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = \phi(x) \geq 0, v(0,x) = \psi(x) \geq 0, & x \in \Omega, \end{cases} \tag{3}$$

where $\Omega \subset R^N (N \leq 3)$ is a bounded domain with smooth boundary $\partial\Omega$. ν is the outward unit normal vector of the boundary $\partial\Omega$, the initial data $\phi(x)$ and $\psi(x)$ are nonnegative continuous functions, which are not identically zero. It can be seen that the diffusive flux of prey and predator is respectively

$$J_1 = -\nabla(d_1 u + a_{11}u^2 + a_{12}vu) = -(d_1 + 2a_{11}u + a_{12}v)\nabla u - (a_{12}u)\nabla v,$$
$$J_2 = -\nabla(d_2 v + a_{21}uv + a_{22}v^2) = -(d_2 + a_{21}u + 2a_{22}v)\nabla v - (a_{21}v)\nabla u,$$

d_1 and d_2, a_{11} and a_{22}, and a_{12} and a_{21} are diffusion, self-diffusion, and cross-diffusion coefficients, respectively. d_1 and d_2, a_{11} and a_{22} are positive, a_{12} and a_{21} can be positive or negative. The homogeneous Neumann boundary condition indicates that the system is self-contained with zero population flux across the boundary. In this connection, we must point out, Shigesada, Kawasaki and Teramoto first proposed a strongly coupled reaction-diffusion model with Lotka-Volterra type reaction terms to investigate the more complex ecological phenomenon such as spatial segregation of interacting population species in one-dimensional space[Shigesada *et al.*, 1979]. The readers can also see[Banerjee *et al.*, 2018; Cantrell & Cosner, 2001; Liu *et al.*, 2018; Lou & Ni, 1996; Madzvamuse *et al.*, 2015; Mukherjee*et al.*, 2018; Ni & Tang, 2005; Sun *et al.*, 2012; Tulumello *et al.*, 2014] for this aspect.

For the sake of simplicity, by applying the following scaling:

$$rt \mapsto t,\ \frac{u}{K} \mapsto u,\ v \mapsto v,\ \frac{k}{rbK} \mapsto k,\ \frac{a}{bK} \mapsto a,\ \frac{c}{bK} \mapsto m,\ \frac{s}{r} \mapsto \delta,$$

$$\frac{sh}{rK} \mapsto \beta, \frac{d_1}{K} \mapsto d_1, \frac{a_{11}}{K^2} \mapsto a_{11}, \frac{a_{12}}{K} \mapsto a_{12}, \frac{a_{21}}{K} \mapsto a_{21}.$$

Then system (2) and (3) can be simplified as follows:

$$\begin{cases} \dfrac{du}{dt} = u(1-u) - \frac{kuv}{a+u+mv}, \\ \dfrac{dv}{dt} = v\left(\delta - \frac{\beta v}{u}\right), \end{cases} \tag{4}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta[(d_1 + a_{11}u + a_{12}v)u] = u(1-u) - \frac{kuv}{a+u+mv}, & t > 0,\ x \in \Omega, \\ \dfrac{\partial v}{\partial t} - \Delta[(d_2 + a_{21}u + a_{22}v)v] = v\left(\delta - \frac{\beta v}{u}\right), & t > 0,\ x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = \phi(x) \geq 0, v(0,x) = \psi(x) \geq 0, & x \in \Omega. \end{cases} \tag{5}$$

In the previous studies, just few works such as[Tian *et al.*, 2010; Wen & Fu, 2009] have been concentrated on the occurrence of Turing instability. Thus, the current one is devoted to investigating the effects of self-diffusion and cross-diffusion on Turing instability of predator-prey model. We find that the changeability of self-diffusion and cross-diffusion in any case will not change the stability of the semi-trivial equilibrium point. However, the self-diffusion and cross-diffusion play an important role in the stability of positive equilibrium point, and they either make the original Turing instability continue to exist, or make it disappear and become stable. What's more, in this paper, we also give that the cross-diffusion can produce stationary patterns, that is, strong cross-diffusions are helpful for the appearance of non-constant positive steady states, i.e., the positive solutions for the corresponding elliptic problem. For the evolutionary systems, steady state solutions play an important role in understanding the long time behavior of the corresponding Cauchy type problem. Hence, one of the main goals in this paper is to establish the positive solutions of the following diffusive predator-prey system,

$$\begin{cases} -\Delta[(d_1 + a_{11}u + a_{12}v)u] = u(1-u) - \dfrac{kuv}{a+u+mv}, & x \in \Omega, \\ -\Delta[(d_2 + a_{21}u + a_{22}v)v] = v\left(\delta - \dfrac{\beta v}{u}\right), & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{6}$$

The rest of this paper is organized as follows. In Section 2, we analyze the effects of self-diffusion and cross-diffusion on Turing instability. In Section 3, a priori upper and lower bounds for positive solutions of (6) are given, and then the existence and nonexistence of nonconstant positive solutions of (6) are investigated. In section 4, we present some numerical results of system (5) by taking different values of diffusion, self-diffusion and cross-diffusion coefficents. The paper ends with a brief discussion in section 5.

2. Effects of self- and cross-diffusion

In this section, the effects of self-diffusion and cross-diffusion on the stability of the constant equilibrium state are mainly discussed. By simple calculation, system (5) has two non-trivial spatially uniform equilibria given by $E_1(1,0)$ and $E^*(u^*,v^*)$, where

$$\begin{aligned} u^* &= \tfrac{-(a\beta-\beta-m\delta+k\delta)+\sqrt{(a\beta-\beta-m\delta+k\delta)^2+4a\beta(\beta+m\delta)}}{2(\beta+m\delta)} > 0, \\ v^* &= \tfrac{\delta}{\beta}u^* > 0. \end{aligned}$$

2.1. *Analysis of system (4) for non-trivial equilibrium*

In the following, we first discuss the stability of the equilibrium of the ordinary differential system (4). This also provides a basis for the further reflection of the effects of diffusion, self-diffusion and cross-diffusion.

The Jacobi matrix of system (4) at $E_1(1,0)$ is

$$\begin{pmatrix} -1 & -\frac{k}{a+1} \\ 0 & \delta \end{pmatrix}.$$

The characteristic polynomial of the Jacobi matrix is

$$(\lambda+1)(\lambda-\delta)=0. \tag{7}$$

Obviously, $E_1(1,0)$ is unstable for system (4), and $E_1(1,0)$ is a saddle point.

Next, we discuss the stability of the positive equilibrium point. The Jacobi matrix of system (4) at $E^*(u^*,v^*)$ is

$$\begin{pmatrix} l & -n \\ \frac{\delta^2}{\beta} & -\delta \end{pmatrix},$$

where

$$l = u^*(-1+\frac{kv^*}{(a+u^*+mv^*)^2}), \quad n=\frac{ku^*(a+u^*)}{(a+u^*+mv^*)^2}.$$

The characteristic polynomial of the Jacobi matrix is

$$\lambda^2+(\delta-l)\lambda-l\delta+n\frac{\delta^2}{\beta}=0. \tag{8}$$

If

$$\delta-l>0, -l\delta+n\frac{\delta^2}{\beta}>0,$$

that is, $l<\delta$, $l<n\frac{\delta}{\beta}$, then $E^*(u^*,v^*)$ is stable for system (4). Therefore, when $l<\min\{\delta, n\frac{\delta}{\beta}\}$, we know that the positive equilibrium $E^*(u^*,v^*)$ of system (4) is stable.

2.2. *Analysis of the PDE system*

For the concision and the convenience of calculation below, define the new variables

$$\begin{aligned} w_1 &:= w_1(u,v) = (d_1+a_{11}u+a_{12}v)u, \\ w_2 &:= w_2(u,v) = (d_2+a_{21}u+a_{22}v)v. \end{aligned}$$

Therefore, (5) reduces to the following system

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta w_1 = u(1-u) - \frac{kuv}{a+u+mv}, & t>0, x\in\Omega, \\ \frac{\partial v}{\partial t} - \Delta w_2 = v\left(\delta - \frac{\beta v}{u}\right), & t>0, x\in\Omega, \\ w_1 = (d_1 + a_{11}u + a_{12}v)u, & t>0, x\in\Omega, \\ w_2 = (d_2 + a_{21}u + a_{22}v)v, & t>0, x\in\Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu} = 0, & t>0,\ x\in\partial\Omega \\ u(0,x) = \phi(x) \geq 0, v(0,x) = \psi(x) \geq 0, x\in\Omega. & \end{cases} \tag{9}$$

The linearization of system (9) at $(1, 0, d_1 + a_{11}, 0)$ is

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta w_1 = -u - \frac{k}{a+1}v, & t>0, x\in\Omega, \\ \frac{\partial v}{\partial t} - \Delta w_2 = \delta v, & t>0, x\in\Omega, \\ w_1 = (d_1 + 2a_{11})u + a_{12}v, & t>0, x\in\Omega, \\ w_2 = (d_2 + a_{21})v, & t>0, x\in\Omega. \end{cases} \tag{10}$$

In following, let $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_i \leq \cdots$ be all eigenvalues of the operator $-\Delta$ on Ω with homogeneous Neumann boundary condition, $E(\mu_i)$ is the space of eigenfunctions corresponding to μ_i for $i = 0, 1, 2, \cdots$. Denote $\mathbf{X}_{ij} := \left\{\mathbf{c}\cdot\varphi_{ij} : \mathbf{c}\in\mathbb{R}^2\right\}$, where $\{\varphi_{ij}\}$ are orthonormal basis of $E(\mu_i)$ for $j = 1, 2, \cdots, \dim[E(\mu_i)]$,

$$\mathbf{X} := \left\{\mathbf{u} = (u, v) \in [C^1(\bar{\Omega})]^2 : \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0\right\},$$

and so $\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i$, where $\mathbf{X}_i = \bigoplus_{j=1}^{\dim[E(\mu_i)]} \mathbf{X}_{ij}$.

Evidently, system (10) has non-trivial solutions if and only if

$$\begin{vmatrix} \lambda + 1 + \mu_i(d_1 + 2a_{11}) & \frac{k}{a+1} + \mu_i a_{12} \\ 0 & \lambda - \delta + \mu_i(d_2 + a_{21}) \end{vmatrix} = 0,$$

that is,

$$[\lambda + 1 + \mu_i(d_1 + 2a_{11})][\lambda - \delta + \mu_i(d_2 + a_{21})] = 0.$$

Then, we get $\lambda_1 = -1 - \mu_i(d_1 + 2a_{11}) < 0$, and $\lambda_2 = \delta - \mu_i(d_2 + a_{21})$. Since $\mu_1 = 0$, then $\lambda_2 = \delta > 0$. Hence, $E_1(1, 0)$ is unstable for system (5). Based on the above analysis, we have the following conclusions.

Theorem 1. *Semi-trivial equilibrium $E_1(1, 0)$ is an unstable equilibrium of ordinary differential equations (4), and it is also an unstable steady-state of partial differential equations (5).*

Remark 2.1. According to the formula of $\lambda_2 = \delta - \mu_i(d_2 + a_{21})$, when μ_i is sufficiently large, we can see that $\lambda_2 < 0$, due to the emergence of diffusion d_2 and cross-diffusion a_{21}. Therefore, this results show that although the diffusion d_2 and cross-diffusion a_{21} enable the characteristics roots decline, and can make the characteristic roots be negative for sufficiently large i, but due to $\mu_1 = 0$, they can not as yet change the stability of equilibrium point $E_1(1, 0)$.

Similarly, the linearization of system (9) at (u^*, v^*, w_1^*, w_2^*) is the following system

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta w_1 = -lu - nv, & t>0, x\in\Omega, \\ \frac{\partial v}{\partial t} - \Delta w_2 = \frac{\delta^2}{\beta} - \delta v, & t>0, x\in\Omega, \\ w_1 = (d_1 + 2a_{11}u^* + a_{12}v^*)u + a_{12}u^*v, & t>0, x\in\Omega, \\ w_2 = a_{21}v^*u + (d_2 + a_{21}u^* + 2a_{22}v^*)v, & t>0, x\in\Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu} = 0, & t>0,\ x\in\partial\Omega, \end{cases} \tag{11}$$

where $w_1^* = w_1(u^*, v^*)$, $w_2^* = w_2(u^*, v^*)$. The characteristic roots satisfy

$$\begin{vmatrix} \lambda - l + \mu_i(d_1 + 2a_{11}u^* + a_{12}v^*) & n + \mu_i a_{12}u^* \\ -\frac{\delta^2}{\beta} + \mu_i a_{21}v^* & \lambda + \delta + \mu_i(d_2 + a_{21}u^* + 2a_{22}v^*) \end{vmatrix} = 0.$$

Denote

$$M_i = \begin{pmatrix} l - \mu_i(d_1 + 2a_{11}u^* + a_{12}v^*) & -n - \mu_i a_{12}u^* \\ \frac{\delta^2}{\beta} - \mu_i a_{21}v^* & -\delta - \mu_i(d_2 + a_{21}u^* + 2a_{22}v^*) \end{pmatrix}.$$

Then,

$$\begin{aligned} \det M_i = &\mu_i^2[(d_1 + 2a_{11}u^* + a_{12}v^*)(d_2 + a_{21}u^* + 2a_{22}v^*) - a_{12}a_{21}u^*v^*] \\ &+\mu_i[-l(d_2 + a_{21}u^* + 2a_{22}v^*) + \delta(d_1 + 2a_{11}u^* + a_{12}v^*) + a_{12}u^*\tfrac{\delta^2}{\beta} - na_{21}v^*] \\ &+n\tfrac{\delta^2}{\beta} - l\delta. \end{aligned}$$

Let

$$A_1 = (d_1 + 2a_{11}u^* + a_{12}v^*)(d_2 + a_{21}u^* + 2a_{22}v^*) - a_{12}a_{21}u^*v^*,$$

$$A_2 = -l(d_2 + a_{21}u^* + 2a_{22}v^*) + \delta(d_1 + 2a_{11}u^* + a_{12}v^*) + a_{12}u^*\frac{\delta^2}{\beta} - na_{21}v^*,$$

$$A_3 = n\frac{\delta^2}{\beta} - l\delta.$$

Obviously, $A_1 > 0$. In addition, according to $l < \min\{\delta, n\frac{\delta}{\beta}\}$, we have $A_3 > 0$. So, we get

$$\det M_i = A_1\mu_i^2 + A_2\mu_i + A_3,$$

and

$$\mathrm{trace} M_i = l - \delta - \mu_i(d_1 + 2a_{11}u^* + a_{12}v^* + d_2 + a_{21}u^* + 2a_{22}v^*) < 0.$$

Based on the above definition and the analysis of the characteristic equation, we can know that

(i) If $A_2 \geq 0$, then $\det M_i > 0$. In addition, $\mathrm{trace} M_i < 0$. We know that when $A_2 \geq 0$, the interior equilibrium $E^*(u^*, v^*)$ of system (5) is stable. This indicates that Turing instability does not occur.
(ii) If $A_2 < 0$, $A_2^2 - 4A_1A_3 < 0$, then $\det M_i > 0$. So, the interior equilibrium $E^*(u^*, v^*)$ of system (5) is stable, and Turing instability also does not occur.
(iii) If $A_2 < 0$, $A_2^2 - 4A_1A_3 > 0$, denote $\det M = A_1\mu^2 + A_2\mu + A_3$, where μ is nonnegative real number, then $\min\limits_{\mu} \det M < 0$, and $\det M = 0$ have two real roots,

$$k_{1,2} = \frac{-A_2 \pm \sqrt{A_2^2 - 4A_1A_3}}{2A_1} > 0. \tag{12}$$

Furthermore, if there exists some μ_i such that $0 < k_1 < \mu_i < k_2$, then $\det M_i < 0$, the interior equilibrium $E^*(u^*, v^*)$ of system (5) is unstable. Thus, Turing instability appears.

Therefore, we get the following theorem.

Theorem 2. *Let $l < \min\{\delta, n\frac{\delta}{\beta}\}$. Then the positive equilibrium $E^*(u^*, v^*)$ of ordinary differential equations (4) is stable. Furthermore, if $A_2 \geq 0$ or $A_2 < 0, A_2^2 - 4A_1A_3 < 0$, then the positive equilibrium $E^*(u^*, v^*)$ of partial differential equations (5) is also stable, i.e., Turing instability does not appear; if $A_2 < 0, A_2^2 - 4A_1A_3 > 0$, and there exist some μ_i such that $0 < k_1 < \mu_i < k_2$ where k_1, k_2 are defined by (12), then the positive equilibrium $E^*(u^*, v^*)$ of partial differential equations (5) is unstable, that is, Turing instability occurs.*

Remark 2.2. Since $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_i \leq \cdots$ are eigenvalues of operator $-\Delta$ on Ω with Neumann boundary conditions, according to the nature of the eigenvalues, we know that μ_i is discrete and $\mu_i \to \infty$, when $i \to \infty$. If the size of the domain Ω is changed, the corresponding eigenvalues μ_i are changed continuously. Particularly, when the size of Ω becomes large, μ_i and μ_{i+1} will become very close, so we can find a μ_i such that $k_1 < \mu_i < k_2$ where k_1, k_2 are defined by (12). Conversely, when the size of Ω becomes very small, $\{\mu_i\}_{i=1}^{\infty}$ becomes very discrete, so we can not find a μ_i such that $k_1 < \mu_i < k_2$, and Turing instability does not appear, which indicates that no pattern exists when the spatial domain is sufficiently small.

Below we give four kinds of situation to further analyze the effects of self-diffusion and cross-diffusion.

(i) $a_{11} = a_{21} = a_{22} = 0$, i.e., system (5) has cross-diffusion a_{12}. Firstly, we investigate the effects of diffusion on the stability of system (5) with $a_{11} = a_{12} = a_{21} = a_{22} = 0$. We suppose that $l > 0$ holds. At this time, we have

$$\mathrm{det}M = d_1 d_2 \mu^2 + [-ld_2 + \delta d_1]\mu + n\frac{\delta^2}{\beta} - l\delta.$$

If

$$\delta d_1 < ld_2, \quad 2\sqrt{d_1 d_2 (n\frac{\delta^2}{\beta} - l\delta)} < ld_2 - \delta d_1,$$

then $\mathrm{det}M = 0$ have two positive real roots k_1, k_2. Furthermore, if there exist some μ_i such that $0 < k_1 < \mu_i < k_2$ where k_1, k_2 are defined by (12), then the positive equilibrium is unstable, that is, Turing instability occurs.

However, when $a_{11} = a_{21} = a_{22} = 0$, i.e., system (5) has cross-diffusion a_{12}, then

$$\mathrm{det}M_i = [(d_1 + a_{12}v^*)d_2]\mu_i^2 + [-ld_2 + \delta(d_1 + a_{12}v^*) + a_{12}u^*\frac{\delta^2}{\beta}]\mu_i + n\frac{\delta^2}{\beta} - l\delta.$$

We can see that due to the emergence of a_{12}, when a_{12} is sufficiently large such that

$$-ld_2 + \delta(d_1 + a_{12}v^*) + a_{12}u^*\frac{\delta^2}{\beta} \geq 0,$$

then $\mathrm{det}M_i > 0$. Therefore, besides $\mathrm{trace}M_i < 0$ the positive equilibrium become stable and Turing instability disappears. If a_{12} is sufficiently small, then Turing instability still exists for system (5).

From the above analysis, we can see that when system (5) only has diffusion d_1, d_2, Turing instability occurs. What's more, if system (5) involves cross-diffusion a_{12} and when a_{12} is sufficiently small, Turing instability still exists. However, if a_{12} is sufficiently large, then Turing instability disappears. Therefore, the cross-diffusion a_{12} has positive effect on the stability of the positive equilibrium $E^*(u^*, v^*)$. This indicates that if the prey disperses quickly from a high density domain of the predator to a low density one, then the predator and prey species may coexist in the interacting habitat uniformly.

(ii) $a_{11} = a_{12} = a_{22} = 0$, i.e., system (5) has cross-diffusion a_{21}. Thus,

$$\mathrm{det}M_i = [d_1(d_2 + a_{21}u^*)]\mu_i^2 + [-l(d_2 + a_{21}u^*) + \delta d_1 - na_{21}v^*]\mu_i + n\frac{\delta^2}{\beta} - l\delta. \tag{13}$$

Correspondingly,

$$A_1 = d_1(d_2 + a_{21}u^*) > 0, \quad A_2 = -ld_2 + \delta d_1 - a_{21}(lu^* + nv^*), \quad A_3 = n\frac{\delta^2}{\beta} - l\delta > 0.$$

If $lu^* + nv^* < 0$, then the positive constant steady state $E^*(u^*, v^*)$ is stable when cross-diffusion a_{21} is sufficiently large such that the coefficient of μ_i in (13) is not less than zero.

If $lu^* + nv^* > 0$, when cross-diffusion a_{21} is sufficiently large such that $A_2 < 0, A_2^2 > 4A_1A_3$ and if there exists some μ_i such that $k_1 < \mu_i < k_2$ where k_1, k_2 are given in (12), then the positive constant steady state $E^*(u^*, v^*)$ is unstable. Thus, Turing instability occurs. In particular, if the size of Ω is so large that the eigenvalues μ_i and μ_{i+1} are very close, we can surely find the required μ_i.

From the viewpoint of biology, if the predator disperses quickly from a high density domain of prey to a low density one, it will yield a uniform or nonuniform distribution of the species. This indicates that if the predator disperses quickly from a high density domain of the prey to a low density one, then the predator and prey species may coexist in the interacting habitat uniformly.

(iii) $a_{12} = a_{21} = a_{22} = 0$, i.e., system (5) has self-diffusion a_{11}. In this case,

$$\det M_i = [d_2(d_1 + 2a_{11}u^*)]\mu_i^2 + [-ld_2 + \delta(d_1 + 2a_{11}u^*)]\mu_i + n\frac{\delta^2}{\beta} - l\delta.$$

If a_{11} is large enough such that $-ld_2+\delta(d_1+2a_{11}u^*) \geq 0$, then $\det M_i > 0$. Therefore, the positive constant steady state $E^*(u^*, v^*)$ is stable. Thus, Turing instability does not occur. According to the case **(i)**, when system (5) only has diffusion d_1, d_2, Turing instability occurs. Here, when self-diffusion a_{11} appears and a_{11} is small enough, then Turing instability still exists. However, if a_{11} is large enough, then Turing instability disappears.

This indicates that if the prey disperses quickly from a high density domain of the oneself to a low density one, then the predator and prey species may coexist in the interacting habitat uniformly.

(iv) $a_{11} = a_{12} = a_{21} = 0$, i.e., system (5) has cross-diffusion a_{22}. In this case,

$$\det M_i = [d_1(d_2 + 2a_{22}v^*)]\mu_i^2 + [-l(d_2 + 2a_{22}v^*) + \delta d_1]\mu_i + n\frac{\delta^2}{\beta} - l\delta.$$

Correspondingly,

$$A_1 = d_1(d_2 + 2a_{22}v^*) > 0, \quad A_2 = -l(d_2 + 2a_{22}v^*) + \delta d_1, \quad A_3 = n\frac{\delta^2}{\beta} - l\delta > 0.$$

If $l \leq 0$, then $A_2 \geq 0$, and thus the positive constant steady state $E^*(u^*, v^*)$ is stable.

If $l > 0$, a_{22} is large enough such that $A_2 < 0, A_2^2 > 4A_1A_3$, and there exists some μ_i such that $k_1 < \mu_i < k_2$, where k_1, k_2 are given in (12), then the positive constant steady state $E^*(u^*, v^*)$ is unstable. Similarly, it will yield a uniform or nonuniform distribution of the species.

Based on the above discussions, we have the following results.

Theorem 3.

(i) *Assume that* $a_{11} = a_{21} = a_{22} = 0$. *Then, there exists* $a_{12}^* > 0$ *such that* $E^*(u^*, v^*)$ *is stable for system* (5) *when* $a_{12} \geq a_{12}^*$.

(ii) *Assume that* $a_{11} = a_{12} = a_{22} = 0$. *If* $lu^* + nv^* < 0$, *then there exists* $a_{21}^* > 0$, *such that* $E^*(u^*, v^*)$ *is stable for system* (5) *when* $a_{21} \geq a_{21}^*$. *If* $lu^* + nv^* > 0$, *and* a_{21} *is large enough, and there exists some* μ_i *such that* $0 < k_1 < \mu_i < k_2$, *where* k_1, k_2 *are defined in (12), then the constant positive steady state* $E^*(u^*, v^*)$ *is unstable for system* (4), *Turing instability occurs.*

(iii) *Assume that* $a_{12} = a_{21} = a_{22} = 0$. *Then there exists* $a_{11}^* > 0$ *such that* $E^*(u^*, v^*)$ *is stable for system* (5) *when* $a_{11} \geq a_{11}^*$.

(iv) *Assume that* $a_{11} = a_{12} = a_{21} = 0$, *if* $l \leq 0$, *then* $E^*(u^*, v^*)$ *is stable for system* (5). *If* $l > 0$, *and* a_{22} *is large enough, and there exists some* μ_i *such that* $0 < k_1 < \mu_i < k_2$, *where* k_1, k_2 *are defined in (12), then the constant positive steady state* $E^*(u^*, v^*)$ *is unstable for system* (5), *Turing instability occur.*

3. Nonconstant positive steady states

In this section, we will discuss the non-existence and existence of non-constant positive steady states of system (6). That is, we will give that the cross-diffusion can produce stationary patterns. Specifically, strong cross-diffusions are helpful for the appearance of the non-constant positive steady states. Firstly, we give a priori upper and lower bounds for positive solutions of system (6). For this, we need the following two lemmas.

Lemma 1. *[Lou & Ni, 1996] Suppose that* $g \in C(\overline{\Omega}, \mathbb{R})$.

(i) *If* $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *satisfies*

$$\Delta w(x) + g(x, w(x)) \geq 0, \quad x \in \Omega, \quad \frac{\partial w}{\partial \nu} \leq 0, \quad x \in \partial\Omega,$$

and $w(x_0) = \max_{\overline{\Omega}} w$, *then* $g(x_0, w(x_0)) \geq 0$.

(ii) *If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$\Delta w(x) + g(x, w(x)) \leq 0, \quad x \in \Omega, \quad \frac{\partial w}{\partial \nu} \geq 0, \quad x \in \partial\Omega,$$

and $w(x_0) = \min_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \leq 0$.

Lemma 2. *[Lin et al., 1988] Let $c \in C(\overline{\Omega})$, and $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution to*

$$\Delta w(x) + c(x)w(x) = 0, \quad x \in \Omega, \quad \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega.$$

Then there exists a positive constant $C^* = C^*(\Omega, \| c \|_\infty)$ such that

$$\max_{\overline{\Omega}} w \leq C^* \min_{\overline{\Omega}} w.$$

We assume that the classical solutions are in $C^2(\Omega) \cap C^1(\overline{\Omega})$. For notational convenience, we write $\Lambda = \Lambda(a, k, m, \delta, \beta)$ in the sequel. The results of upper and lower bounds can be stated as follows.

Theorem 4. *(Upper bounds) Suppose that D is an arbitrary fixed positive number and $k > m$. If $\frac{a_{ij}}{d_i} \leq D (i, j = 1, 2)$, there exist positive numbers $C_i = C_i(D, \Lambda), i = 1, 2,$ such that*

$$\max_{\overline{\Omega}} u \leq C_1, \quad \max_{\overline{\Omega}} v \leq C_2,$$

for any positive solution (u, v) of (6).

Proof. Assume there is $x_1 \in \overline{\Omega}$ such that $w_1(x_1) = \max_{\overline{\Omega}} w_1$, where $w_1 = (d_1 + 2a_{11}u^* + a_{12}v^*)u + a_{12}u^*v$. By the application of the maximum principle, it yields $u(x_1) \leq 1$, $v(x_1) \leq \frac{a+1}{k-m}$, then

$$\begin{aligned} \max_{\overline{\Omega}} u \leq \frac{1}{d_1} \max_{\overline{\Omega}} w_1 &= \frac{1}{d_1}(d_1 + a_{11}u(x_1) + a_{12}v(x_1))u(x_1) \\ &\leq (1 + \frac{a_{11}}{d_1} + \frac{a_{12}}{d_1}\frac{a+1}{k-m}) \\ &\triangleq C_1(D, \Lambda). \end{aligned} \tag{14}$$

Let $x_2 \in \overline{\Omega}$ be a point such that $w_2(x_2) = \max_{\overline{\Omega}} w_2$, where $w_2 = a_{21}v^*u + (d_2 + a_{21}u^* + 2a_{22}v^*)v$. Therefore, applying Lemma 1, we get $v(x_2) \leq \frac{\delta}{\beta} C_1$ and

$$\begin{aligned} \max_{\overline{\Omega}} v \leq \frac{1}{d_2} \max_{\overline{\Omega}} w_2 &= \frac{1}{d_2}\left(d_2 + a_{21}u(x_2) + a_{22}v(x_2)\right) v(x_2) \\ &\leq \left(1 + \frac{a_{21}}{d_2}C_1 + \frac{a_{22}}{d_2}\frac{\delta}{\beta}C_1\right)\frac{\delta}{\beta}C_1 \\ &\triangleq C_2(D, \Lambda). \end{aligned} \tag{15}$$

This completes the proof. ■

Theorem 5. *(Lower bounds) Suppose $k > m, d_j \geq \varepsilon (j = 1, 2)$, $\frac{a_{ij}}{d_i} \leq D (i, j = 1, 2)$. Thus there exist positive numbers $c_i = c_i(D, \varepsilon, \Lambda)$ such that*

$$\min_{\overline{\Omega}} u \geq c_1, \quad \min_{\overline{\Omega}} v \geq c_2$$

for a arbitrarily positive solution (u, v) of (6).

Proof. Let

$$e_1(x) = \frac{1}{(d_1 + a_{11}u + a_{12}v)}\left(1 - u - \frac{sv}{m + u + nv}\right),$$

$$e_2(x) = \frac{1}{(d_2 + a_{21}u + a_{22}v)} \left(\frac{ru}{m+u+nv} - qr \right).$$

Then

$$\begin{cases} \Delta w_1 + e_1(x) w_1 = 0, x \in \Omega, \partial_\nu w_1 = 0, x \in \partial\Omega, \\ \Delta w_2 + e_2(x) w_2 = 0, x \in \Omega, \partial_\nu w_2 = 0, x \in \partial\Omega. \end{cases} \tag{16}$$

From (14) and (15), there exists a positive constant $\overline{C} \triangleq \overline{C}(D, \varepsilon, \Lambda)$ such that

$$\|e_1(x)\|_\infty, \|e_2(x)\|_\infty \leq \overline{C}.$$

By Harnack inequality, positive constants $\overline{M}_i = \overline{M}_i(\Omega, D, \varepsilon, \Lambda)$ can be obtained, such that

$$\max_{\overline{\Omega}} w_i \leq \overline{M}_i \min_{\overline{\Omega}} w_i, \quad i = 1, 2.$$

Thus,

$$\frac{\max_{\overline{\Omega}} u}{\min_{\overline{\Omega}} u} \leq \overline{M}_1 \left(1 + \frac{a_{11}C_1}{d_1} + \frac{a_{12}C_2}{d_1} \right) \triangleq M_1^*.$$

In the same way, we get

$$\frac{\max_{\overline{\Omega}} v}{\min_{\overline{\Omega}} v} \leq \overline{M}_2 \left(1 + \frac{a_{21}C_1}{d_2} + \frac{a_{22}C_2}{d_2} \right) \triangleq M_2^*.$$

Now, suppose that Theorem 5 is not true, then there is a sequence $\{d_{j,i}, a_{1j,i}, a_{2j,i}\}_{i=1}^\infty$ with $d_{j,i} \geq D$, $a_{1j,i} \geq 0$, $a_{2j,i} \geq 0$, $j = 1, 2$, and the positive solution (u_i, v_i) of (6) corresponding to (d_j, a_{1j}, a_{2j})=$(d_{j,i}, a_{1j,i}, a_{2j,i})$, such that

$$\min_{\overline{\Omega}} u_i \to 0 \ or \ \min_{\overline{\Omega}} v_i \to 0 \ as \ i \to \infty, \tag{17}$$

where (u_i, v_i) satisfies

$$\begin{cases} -\Delta[(d_{1,i} + a_{11,i}u_i + a_{12,i}v_i)u_i] = u_i(1 - u_i) - \frac{k u_i v_i}{a + u_i + m v_i}, & x \in \Omega, \\ -\Delta[(d_{2,i} + a_{21,i}u_i + a_{22,i}v_i)v_i] = v_i\left(\delta - \frac{\beta v_i}{u_i}\right), & x \in \Omega, \\ \frac{\partial u_i}{\partial \nu} = \frac{\partial v_i}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{18}$$

Moreover, by integrating the problem (18) in Ω, we have

$$\int_\Omega u_i (1 - u_i - \frac{k v_i}{a + u_i + m v_i}) dx = 0, \tag{19}$$

$$\int_\Omega v_i (\delta - \frac{\beta v_i}{u_i}) dx = 0. \tag{20}$$

In virtue of (20) and (u, v) is a classical positive solution, then there exists $x_i \in \overline{\Omega}$ such that $u_i(x_i) = \frac{\beta}{\delta} v_i(x_i)$ for all $i \geq 1$, which implies that both $u(x_i)$ and $v(x_i)$ converge to zero uniformly on $\overline{\Omega}$ as $i \to \infty$. ■

Next, we will give the results referring non-existence of positive solutions of system (6).

Theorem 6. *Let ε, D be arbitrary positive constants, $a_{12} = a_{21} = 0$, $\mu_2(d_2 + 2c_2 a_{22}) > \delta$, c_2 is the positive lower bound of v. So there presents a positive constant $A_{11} = A_{11}(\Omega, D, \varepsilon, \Lambda)$ (or $D_1 = D_1(\Omega, D, \varepsilon, \Lambda)$) causing that* (5) *has no non-constant positive solution if $a_{11} > A_{11}$ (or $d_1 > D_1$), $d_j \geq \varepsilon$, and $\frac{a_{ij}}{d_i} \leq D (i, j = 1, 2)$.*

Proof. Let $(u, v)^T$ be a positive solution of (6) with $a_{12} = a_{21} = 0$. For any $v \in L^1(\Omega)$, let $\overline{g} = \frac{1}{|\Omega|}\int_\Omega v dx$. Multiplying the two equations in (6) by $u - \overline{u}$ and $v - \overline{v}$, respectively, and integrating over Ω, we have

$$\begin{aligned}\int_\Omega (d_1 + 2a_{11}u)|\nabla u|^2 dx &= \int_\Omega \left(u(1-u) - \frac{kuv}{a+u+mv}\right)(u-\overline{u})dx \\ &\le \int_\Omega \{(u-\overline{u})^2 + 2k|u-\overline{u}||v-\overline{v}|\}dx.\end{aligned}$$

Then, by the $\varepsilon-$Young Inequality, we get

$$\int_\Omega \{(u-\overline{u})^2 + 2k|u-\overline{u}||v-\overline{v}|\}dx \le \int_\Omega \{(1+C(\varepsilon))(u-\overline{u})^2 + \varepsilon(v-\overline{v})^2\}dx,$$

where $C(\varepsilon)$ depends only on $\Lambda, \Omega, \varepsilon$ and D.

Similarly,

$$\begin{aligned}\int_\Omega (d_2 + 2a_{22}v)|\nabla v|^2 dx &\le \int_\Omega \{\frac{\beta \overline{v}^2}{u\overline{u}}(u-\overline{u})(v-\overline{v}) + \delta(v-\overline{v})^2\}dx \\ &\le \int_\Omega \left\{C(\varepsilon)(u-\overline{u})^2 + (\delta+\varepsilon)(v-\overline{v})^2\right\}dx.\end{aligned}$$

Considering the Poincaré inequality, we get

$$\begin{aligned}&\mu_2 \int_\Omega [(d_1 + 2c_1a_{11})(u-\overline{u})^2 + (d_2 + 2c_2a_{22})(v-\overline{v})^2]dx \\ \le &\int_\Omega (d_1 + 2a_{11}c_1)|\nabla u|^2 dx + \int_\Omega (d_2 + 2a_{22}c_2)|\nabla v|^2 dx \\ \le &\int_\Omega \{(1+2C(\varepsilon))(u-\overline{u})^2 + (\delta+2\varepsilon)(v-\overline{v})^2\}dx.\end{aligned}$$

Since $\mu_2(d_2 + 2c_2a_{22}) > \delta$, we can choose a sufficiently small value ε such that $\mu_2(d_2 + 2c_2a_{22}) > \delta + 2\varepsilon$. Thus,

$$\mu_2(d_1 + 2c_1a_{11})\int_\Omega (u-\overline{u})^2 dx \le (1+2C(\varepsilon))\int_\Omega (u-\overline{u})^2 dx,$$

if $a_{11} \ge \frac{1+2C(\varepsilon)-d_1\mu_2}{2c_1\mu_2}$ $\left(\text{or } d_1 > \frac{1+2C(\varepsilon)}{\mu_2} - 2c_1a_{11}\right)$. Therefore, we can conclude that $u \equiv \overline{u}, v \equiv \overline{v}$. Then the proof is completed. ■

Remark 3.1. When $a_{12} = a_{21} = 0$, Theorem 6 indicates that large diffusion or self-diffusion have obstructive effect for the existence of non-constant positive steady states.

In the following, we will concern the effects of cross-diffusion in system (6). For details, we shall develop a general setting using the Leray-Schauder degree theory to establish the existence of positive solutions of the system

$$\begin{cases} -\Delta\Phi(\mathbf{w}) = F(\mathbf{w}), & x \in \Omega, \\ \frac{\partial \mathbf{w}}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{21}$$

where

$$\mathbf{w} = (u, v)^T, \quad F(\mathbf{w}) = (f_1, f_2)^T, \quad \Phi(\mathbf{w}) = (\Phi_1(\mathbf{w}), \Phi_2(\mathbf{w}))^T,$$

$$f_1 = u(1-u) - \frac{kuv}{a+u+mv}, \quad f_2 = v\left(\delta - \frac{\beta v}{u}\right),$$

$$\Phi_1(\mathbf{w}) = (d_1 + a_{11}u + a_{12}v)u, \quad \Phi_2(\mathbf{w}) = (d_2 + a_{21}u + a_{22}v)v.$$

Denote

$$\mathbf{X} = \{\mathbf{w} \in [C(\overline{\Omega})]^2 | \frac{\partial \mathbf{w}}{\partial \nu} = 0 \ on \ \partial\Omega\},$$

$$\mathbf{X}^+ = \{(u, v) \in \mathbf{X} | u > 0, v > 0 \ on \ \overline{\Omega}\},$$

$$\mathcal{B}(C) = \left\{(u, v) \in \mathbf{X} | \frac{1}{C} < u, v < C \ on \ \overline{\Omega}\right\},$$

where C is a positive constant which is confirmed to exist according to Theorems 4 and 5. It is obvious that $\det\Phi_{\mathbf{w}}(\mathbf{w})$ is positive for any non-negative $\mathbf{w}$. Therefore $\Phi_{\mathbf{w}}^{-1}$ exists and $\det\Phi_{\mathbf{w}}^{-1}$ is positive. Meanwhile, (21) is equivalent to the following system

$$G(\mathbf{w}) := \mathbf{w} - (I - \Delta)^{-1}\{[\Phi_{\mathbf{w}}(\mathbf{w})]^{-1}[F(\mathbf{w}) + \nabla \mathbf{w} \Phi_{\mathbf{ww}}(\mathbf{w}) \nabla \mathbf{w}] + \mathbf{w}\} = 0 \text{ in } \mathbf{X}^+, \tag{22}$$

where $(I - \Delta)^{-1}$ is the inverse of $(I - \Delta)$ in $\mathbf{X}$. If $G(\mathbf{w}) \neq 0$ for all $\mathbf{w} \in \partial\mathcal{B}$, the Leray-Schauder $\deg(G(\cdot), 0, \mathcal{B}(C))$ can be well-defined. Among them, $G(\cdot)$ is a compact perturbation of an identity operator I. Moreover, after the computations, we can notice that

$$D_{\mathbf{w}}G(\mathbf{w}^*) = I - (I - \Delta)^{-1}\{[\Phi_{\mathbf{w}}(\mathbf{w}^*)]^{-1} F_{\mathbf{w}}(\mathbf{w}^*) + I\} \text{ in } L(\mathbf{X}, \mathbf{X}).$$

If $D_{\mathbf{w}}G$ is reversible, the index of G at the point $\mathbf{w}^*$ can be defined as index $(G(\cdot), \mathbf{w}^*) = (-1)^r$, where r is the number of negative eigenvalues for $D_{\mathbf{w}}G(\mathbf{w}^*)$. Supposing $G \neq 0$ when $w \in \partial\mathcal{B}(C)$, the degree $\deg(G(\cdot), 0, \mathcal{B}(C))$ is equal to the sum of all the indexes of the solutions for $G = 0$ in $\mathcal{B}(C)$,

In order to calculate r, we employ the eigenspace of $-\Delta$. First, we know X_{ij} is invariant under $D_{\mathbf{w}}G(\mathbf{w}^*)$ for each $i \in N$ and each $j \in [1, \dim E(\mu_i)] \cap N$. Thus, μ is an eigenvalue of $D_{\mathbf{w}}G(\mathbf{w}^*)$ on X_{ij} if and only if it is an eigenvalue of the matrix

$$I - \frac{1}{1 + \mu_i}\{[\Phi_{\mathbf{w}}(\mathbf{w}^*)]^{-1} F_{\mathbf{w}}(\mathbf{w}^*) + I\} = \frac{1}{1 + \mu_i}\{\mu_i I - [\Phi_{\mathbf{w}}(\mathbf{w}^*)]^{-1} F_{\mathbf{w}}(\mathbf{w}^*)\}.$$

Therefore, $D_{\mathbf{w}}G(\mathbf{w}^*)$ is invertible if and only if the matrix

$$I - \frac{1}{1 + \mu_i}\{[\Phi_{\mathbf{w}}(\mathbf{w}^*)]^{-1} F_{\mathbf{w}}(\mathbf{w}^*) + I\}$$

is non-singular for any $i \geq 1$. Denote

$$H(\mu) = H(\mathbf{w}^*, \mu) := \det\{\mu I - [\Phi_{\mathbf{w}}(\mathbf{w}^*)]^{-1} F_{\mathbf{w}}(\mathbf{w}^*)\}.$$

If $H(\mu_i) \neq 0$ for any $i \geq 1$, the number of eigenvalues of $D_{\mathbf{w}}G(\mathbf{w}^*)$ on X_{ij} with negative real parts is odd only for that $H(\mu_i) < 0$. Then, we can obtain the following result.

Theorem 7. *Suppose that the matrix $\mu_i I - [\Phi_{\mathbf{w}}(\mathbf{w}^*)]^{-1} F_{\mathbf{w}}(\mathbf{w}^*)$ is non-singular for any $i \geq 1$. There has*

$$\text{index}(G(\cdot), \mathbf{w}^*) = (-1)^\sigma, \sigma = \sum_{i \geq 1, H(\mu_i) < 0} \dim E(\mu_i).$$

Obviously, $\det[\Phi_{\mathbf{w}}(\mathbf{w}^*)]^{-1} > 0$, and

$$H(\mu) = H(\mathbf{w}^*, \mu) = \det[\Phi_{\mathbf{w}}(\mathbf{w}^*)]^{-1} \det\{\mu \Phi_{\mathbf{w}}(\mathbf{w}^*) - F_{\mathbf{w}}(\mathbf{w}^*)\}.$$

Though the calculation, we can note

$$\det\{\mu \Phi_{\mathbf{w}}(\mathbf{w}^*) - F_{\mathbf{w}}(\mathbf{w}^*)\} = A_2 \mu^2 - A_1 \mu + A_0 \triangleq B(\mu), \tag{23}$$

where

$$A_1 = [-(d_1 + 2a_{11}u^* + a_{12}v^*)\delta + (d_2 + a_{21}u^* + 2a_{22}v^*)l - a_{12}u^* \frac{\delta^2}{\beta} + a_{21}v^* n],$$

$$A_2 = [(d_1 + 2a_{11}u^* + a_{12}v^*)(d_2 + a_{21}u^* + 2a_{22}v^*) - a_{12}a_{21}u^* v^*] > 0,$$

and $A_0 = (n\frac{\delta^2}{\beta} - l\delta) > 0$. Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be the two roots of $B(\mu) = 0$ with $\mathrm{Re}\overline{\mu}_1 \leq \mathrm{Re}\overline{\mu}_2$. Then

$$\overline{\mu}_1\overline{\mu}_2 = \frac{A_0}{A_2} > 0.$$

Additionally,

$$\lim_{a_{21}\to\infty} \frac{B(\mu)}{a_{21}} = \Lambda_2'\mu^2 - \Lambda_1'\mu,$$

where

$$\begin{array}{c} \Lambda_2' = \lim_{a_{21}\to\infty} \frac{A_2}{a_{21}} = (d_1 + 2a_{11}u^*)u^* > 0, \\ \Lambda_1' = \lim_{a_{21}\to\infty} \frac{A_1}{a_{21}} = lu^* + nv^* > 0. \end{array}$$

According to (23), using the previous analysis, we get the following lemma.

Lemma 3. *There exists a positive constant a_{21}^* $(a_{21} \geq a_{21}^*)$, bringing the result that $B(\mu) = 0$ has two positive roots $\overline{\mu}_1$ and $\overline{\mu}_2$ satisfying*

$$\lim_{a_{21}\to\infty} \overline{\mu}_1 = 0, \lim_{a_{21}\to\infty} \overline{\mu}_2 = \frac{lu^* + nv^*}{(d_1 + 2a_{11}u^*)u^*} \triangleq \overline{\mu}' > 0,$$

where

$$\begin{cases} B(\mu) < 0, \ \textit{when}\ \mu \in (\overline{\mu}_1, \overline{\mu}_2), \\ B(\mu) > 0, \ \textit{when}\ \mu \in (-\infty, \overline{\mu}_1) \cup (\overline{\mu}_2, \infty). \end{cases} \tag{24}$$

Now we establish the existence of non-constant positive solutions to (21) with respect to the cross-diffusion coefficient a_{21}, as the other parameters are all fixed positive constants. The result is as follows.

Theorem 8. *If $\overline{\mu}' \in (\mu_n, \mu_{n+1})$ for some $n \geq 1$, and the sum $\sigma_n = \sum_{i=2}^{n} dimE(\mu_i)$ is odd, then there exists a positive constant $\tilde{a}_{21}(\tilde{a}_{21} \geq a_{21}^*)$ such that, if $a_{21} > \tilde{a}_{21}$, problem (21) has at least one non-constant positive solution.*

Proof. According to Lemma 3, if $a_{21} \geq a_{21}^*$, then (24) holds and

$$0 = \mu_1 < \overline{\mu}_1 < \overline{\mu}_2, \quad \overline{\mu}_2 \in (\mu_n, \mu_{n+1}).$$

We shall prove that for all $a_{21} > \tilde{a}_{21}$, (21) has at least one non-constant positive solution. The proof will be fulfilled by contradiction. Suppose on the contrary that the assertion is not true for some $a_{21} = \overline{a}_{21} > \tilde{a}_{21}$. Let a_{21} be fixed as $\overline{a}_{21}$.

For $t \in [0, 1]$, define $d_i(t) = td_i + (1-t)\bar{d}_i$, $a_{ii}(t) = ta_{ii} + (1-t)\bar{a}_{ii}$, i = 1, 2, $a_{12}(t) = ta_{12}$, $a_{21}(t) = t\overline{a}_{21}$, and $a_{11} > A_{11}$. Fix $\bar{d}_1, \bar{d}_2, \bar{a}_{22}$, let $\bar{a}_{11}$ be large enough such that Theorem 6 holds for $a_{12} = a_{21} = 0$,

$$\Phi(\mathbf{w}) = (\Phi_1(\mathbf{w}), \Phi_2(\mathbf{w}))^T = ((d_1(t) + a_{11}(t)u + a_{12}(t)v)u, (d_2(t) + a_{21}(t)u + a_{22}(t)v)v)^T.$$

and then consider the following problem

$$G(t, \mathbf{w}) := \mathbf{w} - (I - \Delta)^{-1}\{[\Phi_{\mathbf{w}}(t, \mathbf{w})]^{-1}[F(\mathbf{w}) + \nabla\mathbf{w}\Phi_{\mathbf{ww}}(t, \mathbf{w})\nabla\mathbf{w}] + \mathbf{w}\} = 0 \tag{25}$$

in $\mathbf{X}^+$. Then $\mathbf{w}$ is a positive solution of (21) if and only if it is a positive steady-state of (25) for $t = 1$. It is obvious that $\mathbf{w}^*$ is the unique constant positive steady-state of (25) for all $t \in [0, 1]$. It is evident that $G(1, \mathbf{w}) = G(\mathbf{w})$. Theorem 6 indicates that $G(0, \mathbf{w}) = 0$ only has the positive constant solution $\mathbf{w}^*$ in X^+. By calculation, we have

$$D_{\mathbf{w}}G(t, \mathbf{w}^*) = I - (I - \Delta)^{-1}\{[\Phi_{\mathbf{w}}(t, \mathbf{w}^*)]^{-1}F_{\mathbf{w}}(\mathbf{w}^*) + I\}.$$

Especially,

$$D_{\mathbf{w}}G(0, \mathbf{w}^*) = I - (I - \Delta)^{-1}\{[\widehat{\Phi}_{\mathbf{w}}(\mathbf{w}^*)]^{-1}F_{\mathbf{w}}(\mathbf{w}^*) + I\},$$

$$D_{\mathbf{w}}G(1, \mathbf{w}^*) = I - (I - \Delta)^{-1}\{[\Phi_{\mathbf{w}}(\mathbf{w}^*)]^{-1}F_{\mathbf{w}}(\mathbf{w}^*) + I\} = D_{\mathbf{w}}G(\mathbf{w}^*),$$

where $\widehat{\Phi}_{\mathbf{w}}(\mathbf{w}^*) = \mathrm{diag}(\bar{d}_1 + 2\bar{a}_{11}u^*, \bar{d}_2 + 2\bar{a}_{22}v^*)$. Moreover, we obtain

$$H(\mu) = \det[\Phi_{\mathbf{w}}(\mathbf{w}^*)]^{-1}B(\mu) \text{ } and \text{ } \det[\Phi_{\mathbf{w}}(\mathbf{w}^*)]^{-1} > 0. \tag{26}$$

For $t = 1$, by Lemma 3 and (26), we get

$$\begin{cases} H(\mu_1) = H(0) > 0, \\ H(\mu_i) < 0, \text{when } 2 \le i \le n, \\ H(\mu_i) > 0, \text{when } i > n. \end{cases} \tag{27}$$

Hence, 0 is not an eigenvalue of the matrix $\mu_i I - [\Phi_{\mathbf{w}}(\mathbf{w}^*)]^{-1}F_{\mathbf{w}}(\mathbf{w}^*)$ for any $i \ge 1$, and

$$\sum_{i>1, H(\mu_i)<0} \dim E(\mu_i) = \sum_{i=2}^{n} \dim E(\mu_i) = \sigma_n$$

is odd.

On the side, according to Theorem 7, we get

$$\mathrm{index}(G(1,\cdot), \mathbf{w}^*) = (-1)^r = (-1)^{\sigma_n} = -1. \tag{28}$$

For $t = 0$, we have $H(\mu_i) > 0$ for all $i \ge 1$ as $\bar{a}_{11}$ is large enough. Similarly, we have

$$\mathrm{index}(G(0,\cdot), \mathbf{w}^*) = (-1)^0 = 1. \tag{29}$$

According to Theorems 4 and 5, there has a positive constant C such that the positive solution of (25) satisfies $\frac{1}{C} < u, v < C$. Therefore, $G(t; \mathbf{w}) \ne 0$ for any t ($t \in [0, 1]$) when $\mathbf{w} \in \partial\mathcal{B}(C)$. On the basis of the homotopy invariance of the topological degree, we have

$$\deg(G(1,\cdot), 0, \mathcal{B}(C)) = \deg(G(0,\cdot), 0, \mathcal{B}(C)). \tag{30}$$

In addition, $G(1, \mathbf{w}) = 0$ and $G(0, \mathbf{w}) = 0$ have only the constant positive solution $\mathbf{w}^*$ in $\mathcal{B}(C)$. By (28) and (29), we get

$$\begin{aligned} \deg(G(1,\cdot), 0, \mathcal{B}(C)) &= \mathrm{index}(G(1,\cdot), \mathbf{w}^*) = -1, \\ \deg(G(0,\cdot), 0, \mathcal{B}(C)) &= \mathrm{index}(G(0,\cdot), \mathbf{w}^*) = 1, \end{aligned}$$

which contradicts (30). The proof is completed. ■

4. Numerical Simulations

In this section, using the method of difference method, we present some numerical results of system (5) with homogeneous Neumann boundary conditions on one dimensional spatial domain by choosing different values of diffusion, self-diffusion and cross-diffusion.

From Theorem 2, we know that system (5) has a unique constant positive steady state $E^* = (0.3698, 0.6472)$ when $k = 1, a = 0.01, m = 1, \delta = 3.5, \beta = 2$. By Theorem 2, if $A_2 \ge 0$ or $A_2 < 0, A_2^2 - 4A_1A_3 < 0$, then the constant positive steady state $E^* = (0.3698, 0.6472)$ of (5) is stable, see Figs.1-3(where Fig.1: $a_{11} = a_{12} = a_{21} = a_{22} = 0$, $d_1 = 1, d_2 = 2, u_0 = 0.3000, v_0 = 0.6500$. Fig.2: $a_{11} = 1, a_{12} = 2, a_{21} = 1, a_{22} = 1$, $d_1 = 1, d_2 = 2, u_0 = 0.3000, v_0 = 0.6500$. Fig.3: $a_{11} = 1, a_{12} = -2, a_{21} = 1, a_{22} = 1$, $d_1 = 1, d_2 = 2, u_0 = 0.3000, v_0 = 0.6500$).

If $A_2 < 0, A_2^2 - 4A_1A_3 > 0$, then $E^* = (0.3698, 0.6472)$ of (5) is unstable, that is, Turing instability occurs, see Fig.4(where $a_{11} = 1, a_{12} = -2.13, a_{21} = 1, a_{22} = 1$, $d_1 = 1, d_2 = 2, u_0 = 0.3000, v_0 = 0.6500$).

From Theorem 3, we know that system (5) has a unique constant positive steady state $E^* = (0.0050, 0.0151)$ when $k = 1, a = 0.01, m = 0.01, \delta = 6, \beta = 2$.

If $a_{11} = a_{12} = a_{21} = a_{22} = 0, d_1 = 1, d_2 = 2500$, then $E^* = (0.0050, 0.0151)$ is unstable, see Fig.5(where $a_{11} = a_{12} = a_{21} = a_{22} = 0$, $d_1 = 1, d_2 = 2500, u_0 = 0.0050, v_0 = 0.0150$). However, by Theorem 3, if we assume $a_{11} = a_{21} = a_{22} = 0$, then there exists $a_{12}^* > 0$ such that $E^* = (0.0050, 0.0151)$ is stable for system (5) when $a_{12} \ge a_{12}^*$, see Fig.6(where $a_{11} = a_{21} = a_{22} = 0, a_{12} = 1000$, $d_1 = 1, d_2 = 2500, u_0 = 0.0050, v_0 = 0.0150$).

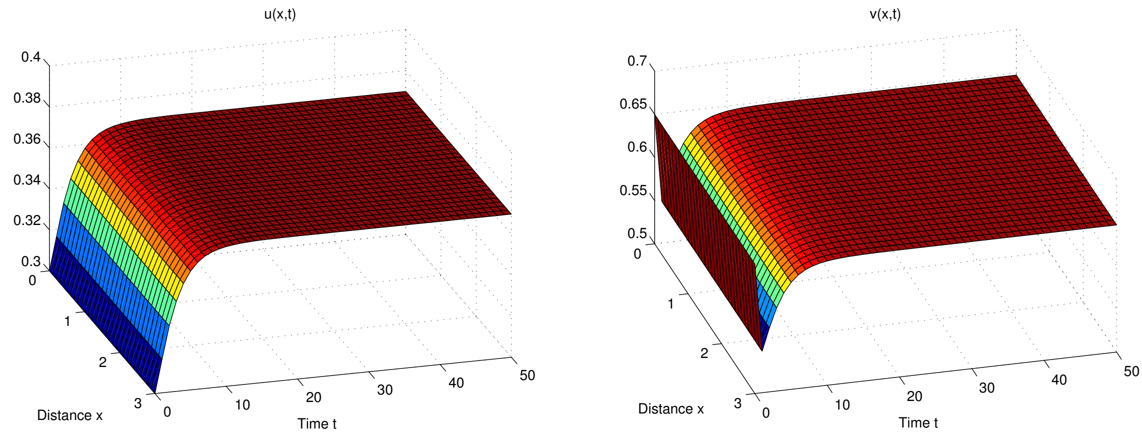

Fig. 1. The solutions $u(x,t)$ and $v(x,t)$ of system converge to the equilibrium values in (0.3698) and (0.6472), respectively, when $a_{11} = a_{12} = a_{21} = a_{22} = 0$, $d_1 = 1, d_2 = 2, u_0 = 0.3000, v_0 = 0.6500$.

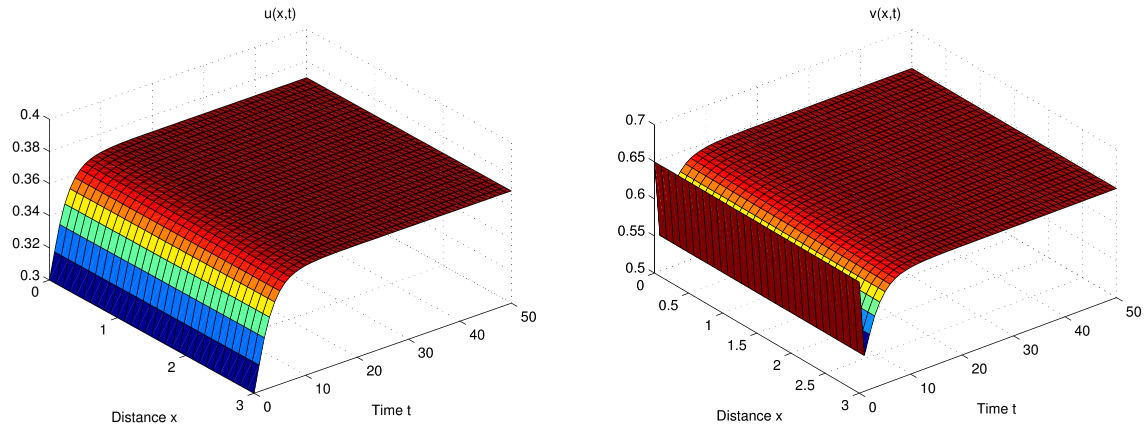

Fig. 2. The solutions $u(x,t)$ and $v(x,t)$ of system converge to the equilibrium values in (0.3698) and (0.6472), respectively, when $a_{11} = 1, a_{12} = 2, a_{21} = 1, a_{22} = 1$, $d_1 = 1, d_2 = 2, u_0 = 0.3000, v_0 = 0.6500$.

Assume $a_{11} = a_{12} = a_{22} = 0$. If $lu^* + nv^* < 0$ $(k = 1, a = 0.01, m = 1, \delta = 3.5, \beta = 2)$, then there exists $a_{21}^* > 0$, such that the constant positive steady state $E^* = (0.3698, 0.6472)$ is stable for system (5) when $a_{21} \geq a_{21}^*$, see Fig.7(where $a_{11} = a_{12} = a_{22} = 0, a_{21} = 100$, $d_1 = 1, d_2 = 2, u_0 = 0.3000, v_0 = 0.6500$).

If $lu^* + nv^* > 0$ $(k = 1, a = 0.01, m = 0.01, \delta = 6, \beta = 2)$, and a_{21} is large enough, then the constant positive steady state $E^* = (0.0050, 0.0151)$ is unstable for system (4), Turing instability occurs, see Fig.8(where $a_{11} = a_{12} = a_{22} = 0, a_{21} = 100$, $d_1 = 1, d_2 = 2500, u_0 = 0.0050, v_0 = 0.0150$).

Assume $a_{12} = a_{21} = a_{22} = 0$. Then there exists $a_{11}^* > 0$ such that the constant positive steady state $E^* = (0.0050, 0.0151)$ is stable for system (5) when $a_{11} \geq a_{11}^*$, see Fig.9(where $a_{21} = a_{12} = a_{22} = 0, a_{11} = 10000$, $d_1 = 1, d_2 = 2500, u_0 = 0.0050, v_0 = 0.0150$).

Assume $a_{11} = a_{12} = a_{21} = 0$, if $l \leq 0$ $(k = 1, a = 0.01, m = 1, \delta = 3.5, \beta = 2)$, then the constant positive steady state $E^* = (0.3698, 0.6472)$ is stable for system (5), see Fig.10(where $a_{11} = a_{12} = a_{21} = 0, a_{22} = 1$, $d_1 = 1, d_2 = 2, u_0 = 0.3000, v_0 = 0.6500$).

If $l > 0$ $(k = 1, a = 0.01, m = 0.01, \delta = 6, \beta = 2)$, and a_{22} is large enough, then the constant positive steady state $E^* = (0.0050, 0.0151)$ is unstable for system (5), Turing instability occur, see Fig.11(where $a_{11} = a_{12} = a_{21} = 0, a_{22} = 100$, $d_1 = 1, d_2 = 2500, u_0 = 0.0050, v_0 = 0.0150$).

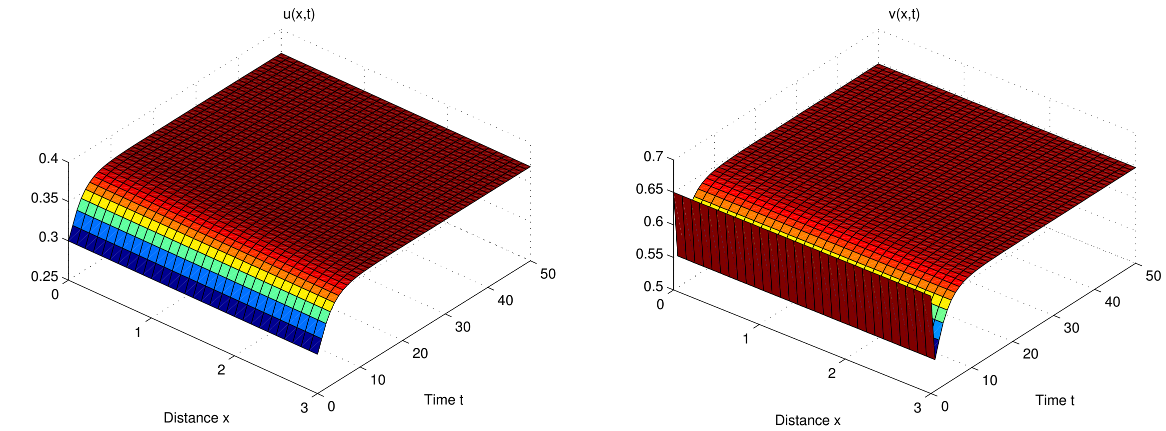

Fig. 3. The solutions $u(x,t)$ and $v(x,t)$ of system converge to the equilibrium values in (0.3698) and (0.6472), respectively, when $a_{11} = 1, a_{12} = -2, a_{21} = 1, a_{22} = 1,\ d_1 = 1, d_2 = 2, u_0 = 0.3000, v_0 = 0.6500$.

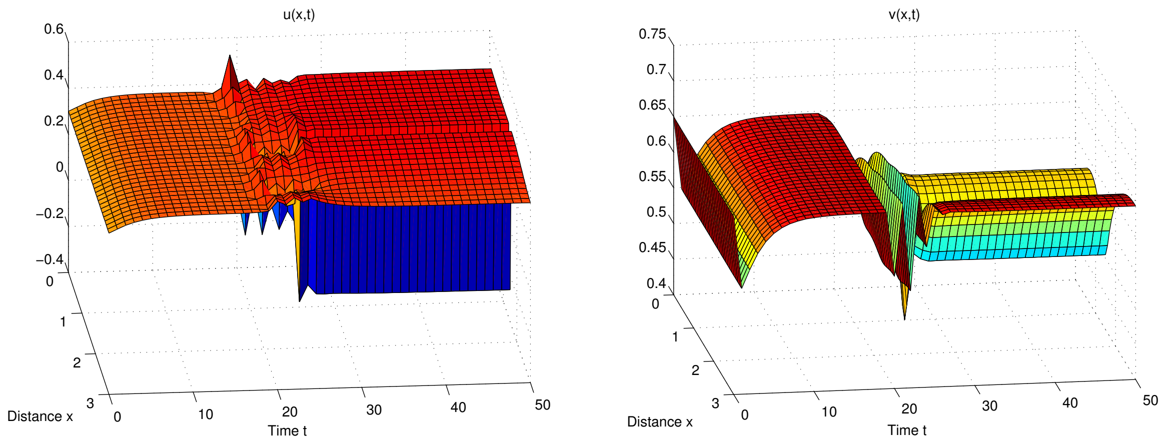

Fig. 4. Numerical simulations of Turing instability for system (5) when $a_{11} = 1, a_{12} = -2.13, a_{21} = 1, a_{22} = 1,\ d_1 = 1, d_2 = 2, u_0 = 0.3000, v_0 = 0.6500$.

5. Conclusions

In this paper, we study a nonlinearly coupled predator-prey diffusion system under homogeneous Neumann boundary condition. The stability of the constant equilibria is discussed. Especially, we found that the changeability of self-diffusion and cross-diffusion in any case will not change the stability of the semi-trivial equilibrium point. For the positive equilibrium point, we find that the self-diffusion and cross-diffusion play an important role, that is, they either make the original Turing instability continue to exist, or make it disappear, and become stable. In addition, Turing instability in standard reaction-diffusion equations is one of the best understood and most widely applicable mechanisms for pattern formation. Here, we establish the existence and nonexistence of stationary pattern formation of this system. The results indicate that large diffusion and self-diffusion have the obstructive effect for the existence of non-constant positive steady-states, while large cross-diffusion have the benefit for existence of stationary pattern. This indicates that if the predator(prey) disperses quickly from a high density domain of the prey(predator) to a low density one, then the predator and prey species may coexist in the interacting habitat uniformly. Meanwhile, if predator(prey) disperses quickly from a high density domain of the oneself to low density one, then the predator and prey species may coexist in the interacting habitat uniformly.

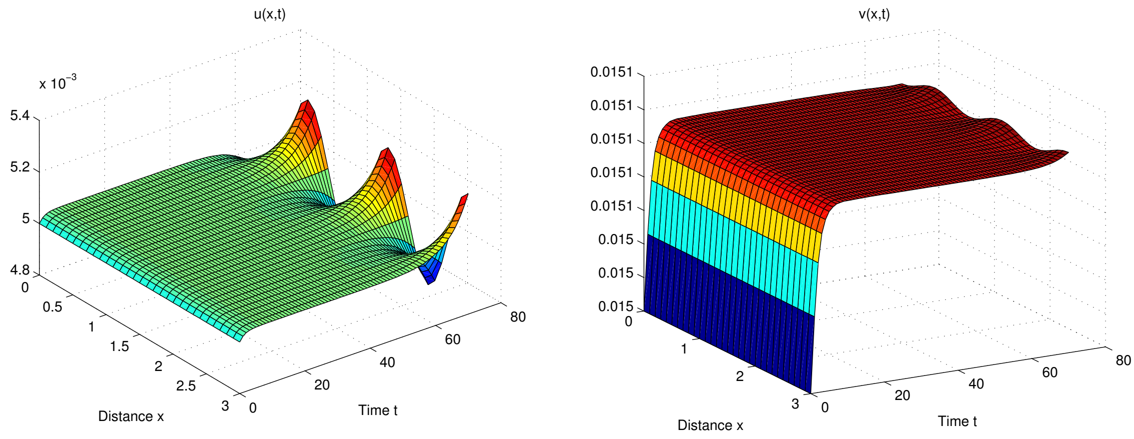

Fig. 5. Numerical simulations of Turing instability for system (5) when $a_{11} = a_{12} = a_{21} = a_{22} = 0$, $d_1 = 1, d_2 = 2500, u_0 = 0.0050, v_0 = 0.0150$.

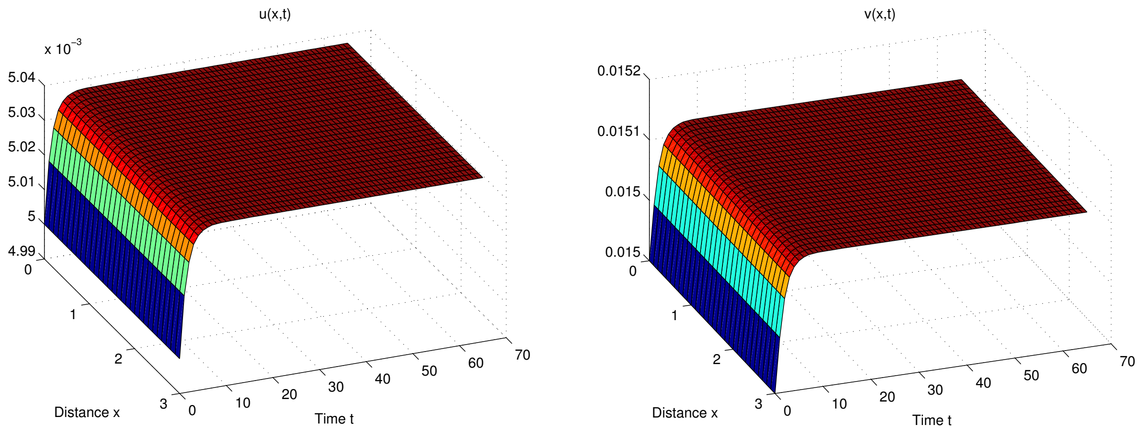

Fig. 6. The solutions $u(x,t)$ and $v(x,t)$ of system converge to the equilibrium values in (0.0050) and (0.0151), respectively, when $a_{11} = a_{21} = a_{22} = 0, a_{12} = 1000$, $d_1 = 1, d_2 = 2500, u_0 = 0.0050, v_0 = 0.0150$.

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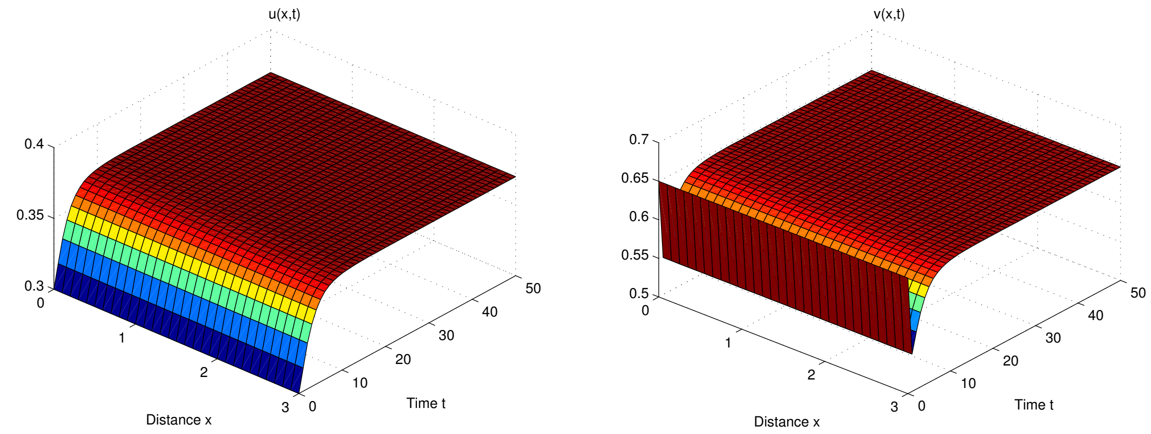

Fig. 7. The solutions $u(x,t)$ and $v(x,t)$ of system converge to the equilibrium values in (0.3698) and (0.6472), respectively, when $a_{11} = a_{12} = a_{22} = 0, a_{21} = 100,\ d_1 = 1, d_2 = 2, u_0 = 0.3000, v_0 = 0.6500$.

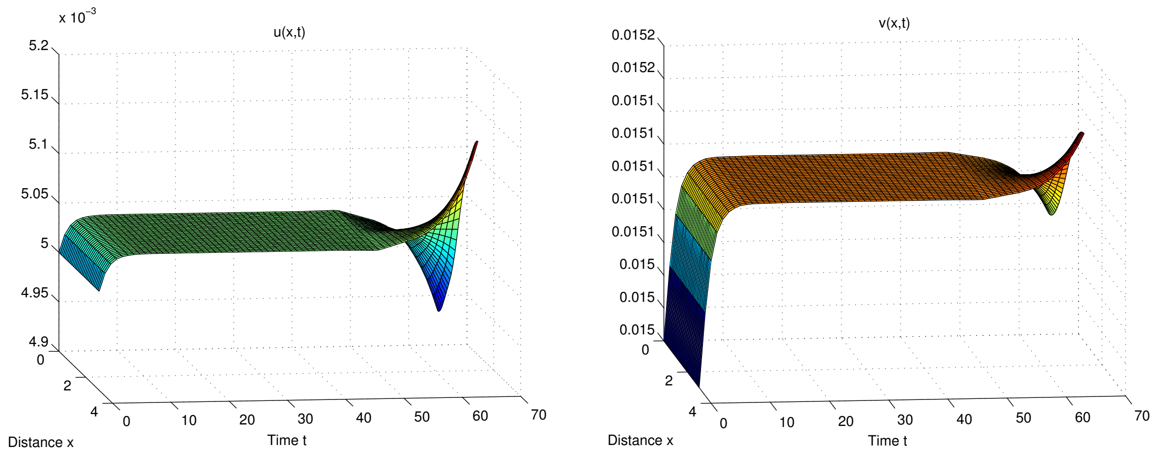

Fig. 8. Numerical simulations of Turing instability for system (5) when $a_{11} = a_{12} = a_{22} = 0, a_{21} = 100,\ d_1 = 1, d_2 = 2500, u_0 = 0.0050, v_0 = 0.0150$.

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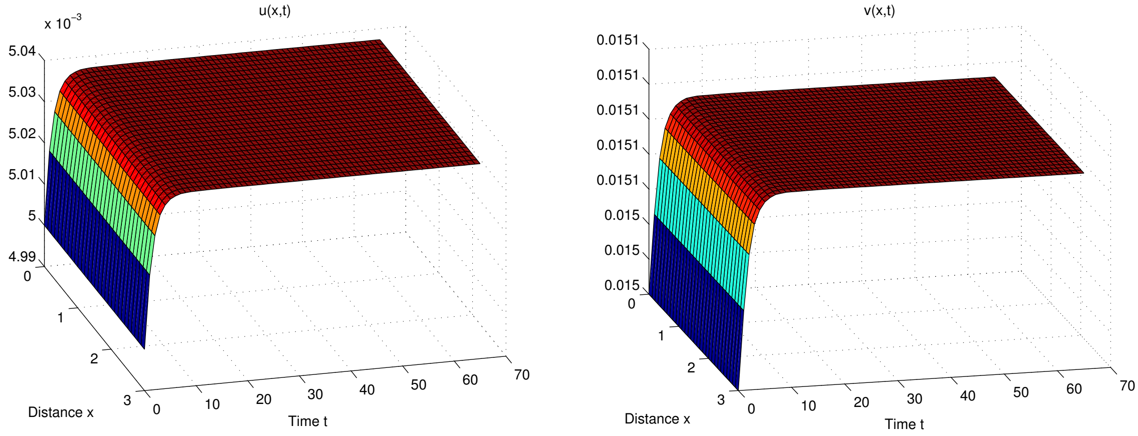

Fig. 9. The solutions $u(x,t)$ and $v(x,t)$ of system converge to the equilibrium values in (0.0050) and (0.0151), respectively, when $a_{21} = a_{12} = a_{22} = 0, a_{11} = 10000,\ d_1 = 1, d_2 = 2500, u_0 = 0.0050, v_0 = 0.0150$.

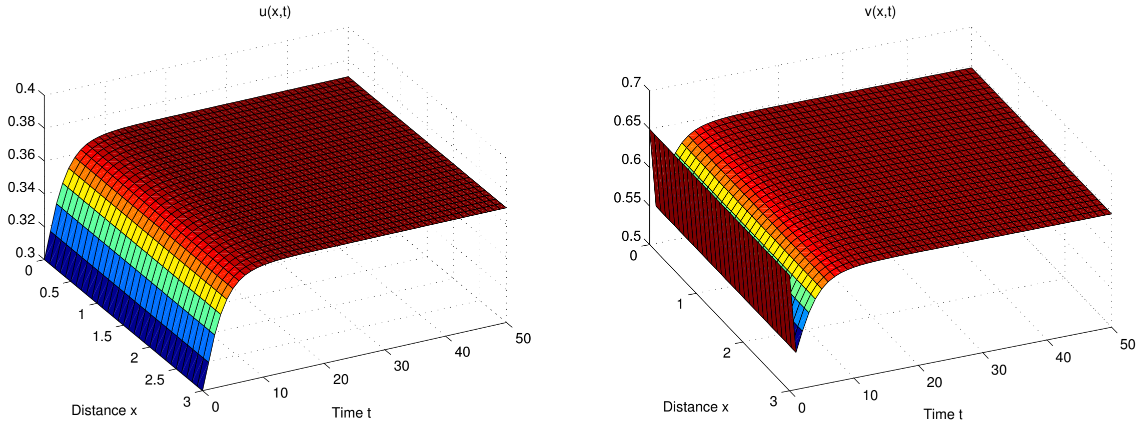

Fig. 10. The solutions $u(x,t)$ and $v(x,t)$ of system converge to the equilibrium values in (0.3698) and (0.6472), respectively, when $a_{11} = a_{12} = a_{21} = 0, a_{22} = 1,\ d_1 = 1, d_2 = 2, u_0 = 0.3000, v_0 = 0.6500$.

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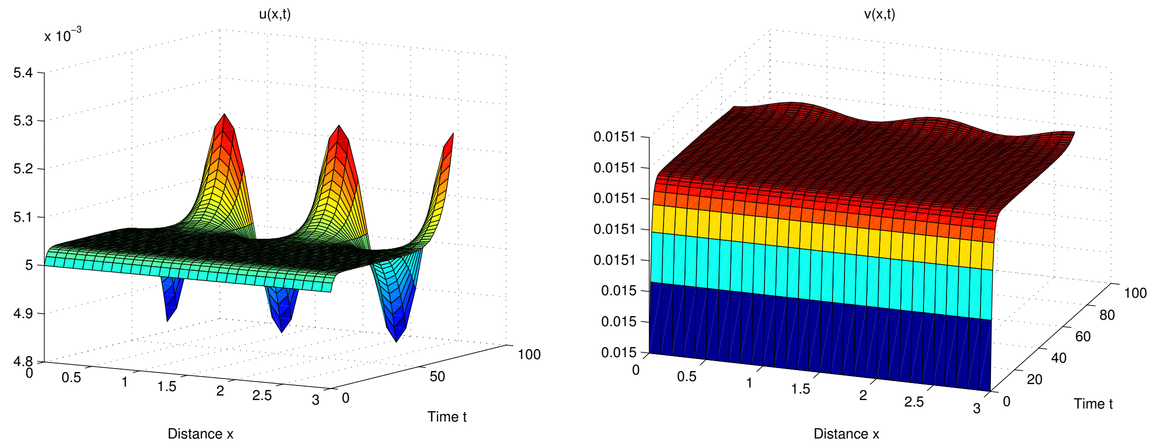

Fig. 11. Numerical simulations of Turing instability for system (5) when $a_{11} = a_{12} = a_{21} = 0, a_{22} = 100$, $d_1 = 1, d_2 = 2500, u_0 = 0.0050, v_0 = 0.0150$.

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